64-3

DWEST

PPLIED

IENCE

ORPORATION

37

N65 13785
(ACCESSION NUMBER)

37
(PAGES)

CR59965
(NASA CR OR TMX OR AD NUMBER)

1
(THRU)

(CODE)

32
(CATEGORY)

FACILITY FORM 602

GPO PRICE $ ________

OTS PRICE(S) $ ________

Hard copy (HC) 2.00

Microfiche (MF) .50

SCIENTIFIC RESEARCH

504-22711

Report No. 64-3

STRESSES AND DISPLACEMENTS IN PRESSURIZED SPHERICAL SHELLS SUBJECTED TO LOCALIZED LOADS

by

John E. Goldberg

and

Dale W. Alspaugh

The Midwest Applied Science Corp.
1220 Potter Drive
McClure Research Park
West Lafayette, Indiana

March 1964

This research was conducted for
The National Aeronautics & Space Administration
under Contract No. NAS8-1674
Huntsville, Alabama

List of Symbols

As a general rule the following conventions will be observed in this report. A superscript (T) will denote a total quantity, i. e. the result of superposing initial pressurization effects and other loading effects. A superscript (n) will denote the n^{th} term of a Fourier expansion of the quantity. (See Art 3.0). A "primed" quantity will denote a nondimensional quantity. There are two exceptions to this last rule, E' and λ. For explanation of these exceptions and the method of nondimensionalization, see Art. 4.2 Certain terms will carry a double sign, e. g., $\pm$, the "upper sign" should be used if the "upper series" expansion of Art. 3.0 is used.

The symbols have the following meanings:

ϕ, θ, z - - -coordinates of a point of the shell; z is measured positive inward from the middle surface, ϕ and θ are the usual angles of a spherical coordinate system.

u, v, w - - -components of the displacement

$N_{\phi\phi}$, $N_{\theta\theta}$, $N_{\phi\theta}$, $N_{\phi z}$, $N_{\theta z}$ - - -internal stress resultants

$M_{\theta\theta}$, $M_{\phi\phi}$, $M_{\theta\phi}$ - - -internal stress couples

X, Y, Z - - -components of surface traction

n - - -denotes the n^{th} term of the expansion

E - - -Young's modulus

ν - - -Poisson's ratio

a - - -undeformed radius of the middle surface

h - - -thickness of the shell

$\tau^{(n)}$ - - -Fourier coefficient of the temperature increase

β - - - Coefficient of thermal expansion.

$$\lambda^{(n)} = \frac{1}{a} \frac{dw^{(n)}}{d\phi}$$

$$\psi_{\phi}^{(n)} = a \sin\phi \, N_{\phi z}^{(n)} \mp n M_{\theta\phi}$$

$$\psi_{\theta}^{(n)} = a \operatorname{Sin}\phi \, N_{\theta\phi}^{(n)} + \operatorname{Sin}\phi \, M_{\theta}^{(n)}$$

$$E' = \frac{Eh}{(1 - \nu^2)} \qquad \qquad \frac{\quad}{n} \int_{-\frac{h}{2}}^{V} \tau^{(n)} \, dz$$

$$D = \frac{Eh^3}{12\,(1 - \nu^2)} \qquad \qquad {}^{(n)} = \frac{a\,(1 + \nu)\,\beta}{h} \int_{-\frac{h}{2}}^{W_2} \tau^{(n)} \, z dz$$

In addition, certain symbols are used which do not appear in the final equations. These symbols are defined at the time of their introduction.

Chapter 1.

1.0 Introduction

In this report a system of ordinary differential equations governing the behavior of spherical shells under arbitrary loads is derived. The resulting system is presented in the chapter as equations 1.1.1 through 1.1.16. The form of these equations is such that a numerical integration technique, e. g. Runge-Kutta, can be easily used. The actual method of solution will be discussed in Chapter 5.

The accuracy of any engineering analysis is limited by the starting equations. The basic equations used in this report are felt to be the most accurate available at present. Linear equations have been used, i. e. products of displacements, etc. are presumed negligible. However, this analysis is designed to include the effects of initial pressurization, and in this sense provides an important extension. In addition, the only further assumptions are those of the validity of Hooke's law and the common Kirchhoff hypothesis of thin shell theory.

Several advantages result from this type of analysis. First, the basic equations are relatively free of simplifying assumptions. Second, very complicated loading systems can be handled. Third, the "output" is in the form of the quantities of most use to the designer, i. e., stress and displacement.

In addition to the assumptions listed above, one additional restriction has been added. The thickness and material properties have been assumed constant. However, it is possible to use the equations 1.1.1-1.1.12 in the special case where the loading is axisymmetric and the thickness and material properties vary only in the direction of the generator. A slight modification of the equations will allow variation of thickness and material properties in the direction of the generator for the case of a nonsymmetric loading.

1.1 First Order Form of Equations

The following eight equations are the final form of the equations which are, for a particular problem, to be integrated numerically. It should be noted that these equations are nondimensional. (See Chapter 4, Section 2.) The effects of temperature gradients are included in these equations.

$$\left.\begin{aligned}\frac{d\,u^{(n)'}}{d\,\phi} &= -\left\{\nu \operatorname{Cot}\phi\right\} u^{(n)'} \mp \left\{\frac{\nu\, n}{\operatorname{Sin}\phi}\right\} \nu^{(n)'} + \left\{1+\nu\right\} w^{(n)'} \\ &\quad + N_{\phi\phi}{}^{(n)'} + T^{(n)'}\end{aligned}\right\} \quad 1.1.1$$

$$\left.\begin{aligned}\frac{d\,\nu^{(n)'}}{d\,\phi} &= \pm\left\{\frac{n}{\operatorname{Sin}\phi}\right\} u^{(n)'} + \left\{\operatorname{Cot}\phi\right\} \nu^{(n)'} \\ &\quad \mp \left\{\frac{2\,n\,D'\operatorname{Cot}\phi}{\operatorname{Sin}\phi\,(1+D')}\right\} w^{(n)'} \pm \left\{\frac{2\,n\,D'}{\operatorname{Sin}\phi\,(1+D')}\right\} \lambda^{(n)} \\ &\quad + \left\{\frac{2}{(1-\nu)(1+D')\operatorname{Sin}\phi}\right\} \psi_{\theta}{}^{(n)'}\end{aligned}\right\} \quad 1.1.2$$

$$\frac{d\,w^{(n)'}}{d\,\phi} = \lambda^{(n)} \qquad 1.1.3$$

$$\left.\begin{aligned}\frac{d\,\lambda^{(n)}}{d\,\phi} &= \left\{\frac{\nu\, n^2}{\operatorname{Sin}^2\phi} - (1+\nu)\right\} w^{(n)'} - \left\{\nu \operatorname{Cot}\phi\right\} \lambda^{(n)} \\ &\quad - N_{\phi\phi}{}^{(n)'} - \left\{\frac{1}{D'}\right\} M_{\phi\phi}{}^{(n)'} - T^{(n)'} - \frac{+^{(n)'}}{D'}\end{aligned}\right\} \quad 1.1.4$$

$$
\begin{aligned}
\frac{d\,N_{\phi\phi}^{(n)'}}{d\,\varphi} = {} & \left\{ (1-\nu^{2}) \operatorname{Cot}\varphi - N'\left(1 \quad \frac{2}{\operatorname{Sin}\varphi}\right) \right\} u^{(n)'} \\
& \pm \left\{ \frac{n \operatorname{Cot}\phi}{\operatorname{Sin}\phi} \left[(1-\nu^{2}) + N' \right] \right\} v^{(n)'} \\
& + \left\{ -(1-\nu^{2}) \operatorname{Cot}\varphi + \frac{2\,n^{2}\,D \operatorname{Cot}\varphi}{\operatorname{Sin}^{2}\varphi\,(1+D')} \left[(1-\nu) - N' \right] \right\} w^{(n)'} \\
& - \left\{ N' + \frac{2\,n^{2}\,D' \left[(1-\nu) - N' \right]}{\operatorname{Sin}^{2}\varphi\,(1+D')} \right\} \lambda^{(n)} - \left\{ (1-\nu) \operatorname{Cot}\phi \right\} N_{\phi\phi}^{(n)'} \\
& + \left\{ \frac{1}{\operatorname{Sin}\phi} \right\} \psi_{\phi}^{(n)'} \pm \left\{ \frac{n}{\operatorname{Sin}^{2}\varphi} \left[1 + \frac{2\,N'}{(1-\nu)(1+D')} \right] \right\} \psi_{\theta}^{(n)'} \\
& - X^{(n)'} - \left\{ (1-\nu) \operatorname{Cot}\phi \right\} T^{(n)'}
\end{aligned}
\tag{1.1.5}
$$

$$
\begin{aligned}
\frac{d\,M_{\phi\phi}^{(n)'}}{d\,\varphi} = {} & -\left\{ (1-\nu^{2})\,D' \operatorname{Cot}^{2}\varphi \right\} u^{(n)'} \mp \left\{ \frac{n\,(1-\nu^{2})\,D' \operatorname{Cot}\varphi}{\operatorname{Sin}\varphi} \right\} v^{(n)'} \\
& + \left\{ \frac{n^{2}\,D'\,(1-\nu) \operatorname{Cot}\phi}{\sin^{2}\phi} \left[\frac{2}{(1+D')} + (1+\nu) \right] \right\} w^{(n)'} \\
& - \left\{ (1-\nu)\,D' \left[\frac{2\,n^{2}}{\sin^{2}\varphi\,(1+D')} + (1+\nu) \operatorname{Cot}^{2}\varphi \right] \right\} \lambda^{(n)} \\
& - \left\{ (1-\nu) \operatorname{Cot}\phi \right\} M_{\phi\phi}^{(n)'} + \left\{ \frac{1}{\operatorname{Sin}\varphi} \right\} \psi_{\phi}^{(n)'} \\
& \pm \left\{ \frac{2\,n\,D'}{\operatorname{Sin}^{2}\phi\,(1+D)} \right\} \psi_{\theta}^{(n)'} - \left\{ (1-\nu) \operatorname{Cot}\phi \right\} +^{(n)'}
\end{aligned}
\tag{1.1.6}
$$

$$d\psi_{\varphi}^{(n)'} = -\left\{(1-\nu)\cos\phi\left[(1+\nu)+N' + \frac{n^2 D'(1+\nu)}{\sin^2\varphi}\right]\right\} u^{(n)'}$$

$$\mp\left\{(1-\nu)\, n\left[(1+\nu)+N' + \frac{n^2 D(1-\nu)}{\sin^2\varphi}\right]\right\} \nu^{(n)'}$$

$$+\left\{\frac{(1-\nu^2)}{\sin\phi}\left[\sin^2\varphi + \frac{n^2 N'}{(1+\nu)} + \frac{n^4 D'}{\sin^2\varphi} + \frac{2n^2 D'\cot^2\varphi}{(1+\nu)(1+D')}\right]\right\} w^{(n)'}$$

$$-\left\{(1-\nu)\cot\phi\left[\frac{n^2 D'(1+\nu)}{\sin\varphi} + \frac{2n^2 D'}{\sin\varphi(1+D')} + N'\sin\varphi\right]\right\}\lambda^{(n)}$$

1.1.7

$$-\{(1+\nu)\sin\phi\}\, N_{\phi\phi}^{(n)'} + \left\{\frac{N'}{D'}\sin\varphi + \frac{\nu n^2}{\sin\varphi}\right\} M_{\phi\phi}^{(n)'}$$

$$\pm\left\{\frac{2nD'\cot\phi}{\sin\phi(1+D')}\right\}\psi_{\theta}^{(n)'} - \{\sin\varphi\}\, Z^{(n)'} + \left\{\frac{N'}{D'}\sin\phi - \frac{n^2(1-2)}{\sin\phi}\right\} +^{(n)'}$$

$$\frac{d\psi_{\theta}^{(n)'}}{d\varphi} = \left\{1 + \frac{1}{\frac{2N'}{(1-\nu)(1+D')}}\right\}\left[\pm\left\{N\cot\varphi\left[vN' + (1-\nu^2)(1+D')\right]\right\} u^{(n)'}\right.$$

$$+\left\{\frac{n^2}{\sin\phi}\left[vN' + (1-\nu^2)(1+D')\right]\right\} v^{(n)'}$$

1.1.8

$$\mp\left\{n\left[vN' + \frac{2N'D'(1+\nu n^2 - \langle 1+\nu\rangle \sin\varphi)}{\sin^2\varphi(1+D')}\right.\right.$$

$$\left.\left.+(1-\nu^2)\left(1+\frac{n^2 D'}{\sin^2\phi}\right)\right]\right\} w^{(n)'}$$

$$\pm\left\{nD'(1+\nu)\cot\phi\left[\frac{2N'}{(1+D')} + (1-\nu)\right]\right\}\lambda^{(n)}$$

$$\mp \{ n [\nu - N' + \frac{2 N' D'}{(1 + D')}] \ldots \text{[illegible]} \ldots \frac{\ldots}{(1 + D')} + \nu] \} M_{\phi\phi}^{(n)'}$$

$$- \{ \mathrm{Cot}\ \phi \} \psi_{\theta}^{(n)'} - \{ \mathrm{Sin}\ \phi \} \Upsilon^{(n)'} \mp \{ - N' n + \frac{2 n' n D'}{(1 + D')}$$

$$- n (1 - \nu) \} T^{(n)'} \pm \{ \frac{2 n N'}{(1 + D')} + n (1 - \nu) \} + ^{(n)'}] \qquad 1.1.8$$

In addition to the eight dependent variables computed above, the following quantities are of interest as "output data".

$$N_{\theta\theta}^{(n)'} = \{ (1 - \nu^2) \mathrm{Cot}\ \phi \} u^{(n)'} \pm \{ \frac{n (1 - \nu^2)}{\mathrm{Sin}\ \phi} \} v^{(n)'}$$

$$- \{ (1 - \nu^2) \} w^{(n)'} + \{ \nu \} N_{\phi\phi}^{(n)'} - (1 - \nu) T^{(n)'} \qquad 1.1.9$$

$$M_{\theta\theta}^{(n)'} = - \{ (1 - \nu^2) D' \mathrm{Cot}\ \phi \} u^{(n)'} \mp \{ \frac{n (1 - \nu^2) D'}{\mathrm{Sin}\ \phi} \} v^{(n)'}$$

$$+ \{ \frac{n^2 (1 - \nu^2) D'}{\mathrm{Sin}^2\ \phi} \} w^{(n)'} - \{ (1 - \nu^2) D' \mathrm{Cot}\ \phi \} \lambda^{(n)} \qquad 1.1.10$$

$$+ \{ \nu \} M_{\phi\phi}^{(n)'} - (1 - \nu) t^{(n)'}$$

$$N_{\theta\phi}^{(n)'} = \mp \{ \frac{n D' (1 - \nu) \mathrm{Cot}\ \phi}{\mathrm{Sin}\ \phi\ (1 + D')} \} w^{(n)'} \pm \{ \frac{n D' (1 - \nu)}{\mathrm{Sin}\ \phi\ (1 + D')} \} \lambda^{(n)}$$

$$+ \{ \frac{1}{\mathrm{Sin}\ \phi\ (1 + D')} \} \psi_{\theta}^{(n)'} \qquad 1.1.11$$

$$M_{\theta\phi}^{(n)'} = \pm \left\{ \frac{n\,D'\,(1-\nu)\,\mathrm{Cot}\,\phi}{\mathrm{Sin}\,\phi\,(1+D')} \right\} w^{(n)'} \mp \left\{ \frac{n\,D'\,(1-\nu)}{\mathrm{Sin}\,\phi\,(1+D')} \right\} \lambda^{(n)} + \left\{ \frac{D'}{\mathrm{Sin}\,\phi\,(1+D')} \right\} \psi_{\theta}^{(n)'} \qquad 1.1.12$$

$$N_{\phi z}^{(n)'} = \left\{ \frac{n^2\,D'\,(1-\nu)\,\mathrm{Cot}\,\phi}{\mathrm{Sin}^2\,\phi\,(1+D')} \right\} w^{(n)'} - \left\{ \frac{n^2\,D'\,(1-\nu)}{\mathrm{Sin}^2\,\phi\,(1+D')} \right\} \lambda^{(n)} \pm \left\{ \frac{n\,D'}{\mathrm{Sin}^2\,\phi\,(1+D')} \right\} \psi_{\theta}^{(n)'} + \left\{ \frac{\cdot}{\mathrm{Sin}\,\phi} \right\} \psi_{\phi}^{(n)'} \qquad 1.1.13$$

$$\begin{aligned} N_{\theta z}^{(n)'} = {} & \pm \left\{ \frac{nD'\,(1-\nu)\,(\mathrm{Cos}^2\,\phi + 1)}{\mathrm{Sin}^3\,\phi\,(1+D')} \right\} w^{(n)'} \\ & \mp \left\{ \frac{2\,n\,D'\,(1-\nu)\,\mathrm{Cot}\,\phi}{\mathrm{Sin}\,\phi\,(1+D')} \right\} \lambda^{(n)} + \left\{ \frac{D'\,\mathrm{Cot}\,\phi}{\mathrm{Sin}\,\phi\,(1+D')} \right\} \psi_{\theta}^{(n)'} \\ & \pm \left\{ \frac{n\,D'\,(1-\nu)}{\mathrm{Sin}\,\phi\,(1+D')} \right\} \frac{d\lambda^{(n)}}{d\phi} - \left\{ \frac{D'}{\mathrm{Sin}\,\phi\,(1+D')} \right\} \frac{d\psi_{\theta}^{(n)'}}{d\phi} \\ & \mp \left\{ \frac{n}{\mathrm{Sin}\,\phi} \right\} M_{\theta\theta}^{(n)'} - \{ 2\,\mathrm{Cot}\,\phi \}\, M_{\theta\phi}^{(n)'} \end{aligned} \qquad 1.1.14$$

where

$$\psi_{\phi}^{(n)'} = \{ \mathrm{Sin}\,\phi \}\, N_{\phi z}^{(n)'} \mp \{ n \}\, M_{\theta\phi}^{(n)'} \qquad 1.1.15$$

$$\psi_{\theta}^{(n)'} = \{ \mathrm{Sin}\,\phi \}\, N_{\theta\phi}^{(n)'} + \{ \mathrm{Sin}\,\phi \}\, M_{\theta\phi}^{(n)'} \qquad 1.1.16$$

Chapter 2.

2.0 Derivation of Equations

The basic equations used in this report are based on the work of Love (A Treatise on the Mathematical Theory of Elasticity, 4th Ed., Dover Publications). The rationale of Love's work will not be discussed; however, it may be pointed out that it is felt that his equations are the best available. One deviation will be made from Love's work. In the expressions for the stress-displacement relations, we use the work of Wang (Applied Elasticity, 1st Ed., McGraw-Hill). It was found that Wang retains one more term than Love. The retention of this extra term seems justified in that other terms of the same order are present in Love's analysis.

Since this analysis includes the effect of initial pressurization of the sphere, the notation $N_{\phi\phi}^{(T)}$ etc. will be used. The superscript (T) indicates that this is the total stress resultant and can be written as

$$N_{\phi\phi}^{(T)} = N_{\phi\phi} + N \qquad 2.0.1$$

where N is the stress resultant due to the initial pressurization and N is the stress resultant caused by loads other than the initial pressure. All other quantities use a similar notation.

2.1 Equilibrium Equations

On page 535, Love gives the equilibrium equations as:

$$\left.\begin{aligned}&\frac{\partial (T_1 B)}{\partial \alpha} - \frac{\partial (S_2 A)}{\partial \beta} - (r_1' S_1 B + r_2' T_2 A)\\ &\qquad + (q_1' N_1 B + q_2' N_2 A) + A B X^{(T)} = 0\end{aligned}\right\} \qquad 2.1.1$$

$$\left.\begin{aligned}&\frac{\partial (S_1 B)}{\partial \alpha} + \frac{\partial (T_2 A)}{\partial \beta} - (p_1' N_1 B + p_2' N_2 A)\\ &\qquad + (r_1' T_1 B - r_2' S_2 A) + ABY^{(T)} = 0\end{aligned}\right\} \qquad 2.1.2$$

$$\left.\begin{aligned}&\frac{\partial (N_1 B)}{\partial \alpha} + \frac{\partial (N_2 A)}{\partial \beta} - (q_1' T_1 B - q_2' S_2 A)\\ &\qquad + (p_1' S_1 B + p_2' T_2 A) + ABZ^{(T)}\end{aligned}\right\} \qquad 2.1.3$$

$$\frac{\partial (H_1 B)}{\partial \alpha} - \frac{\partial (G_2 A)}{\partial \beta} - (G_1 Br_1' + H_2 Ar_2') + N_2 AB = 0 \qquad 2.1.4$$

$$\frac{\partial (G_1 B)}{\partial \alpha} + \frac{\partial (H_2 A)}{\partial \beta} + (H_1 Br_1' - G_2 Ar_2') - N_1 AB = 0 \qquad 2.1.5$$

$$G_1 B p_1' + G_2 A q_2' - (H_1 B_1' - H_2 A p_2') + (S_1 + S_2) AB = 0 \qquad 2.1.6$$

Equations 2.1.1 - 2.1.6 are extremely general. We introduce the following specialization to a spherical shell:

$$\left.\begin{array}{ll} \alpha = \phi & \beta = \theta \\ A = a & B = a \operatorname{Sin} \phi \\ \frac{1}{R_1} = \frac{1}{R_2} = \frac{1}{a} & \\ T_1 = N_{\phi\phi}{}^{(T)} & T_2 = N_{\theta\theta}{}^{(T)} \\ N_1 = N_{\phi z}{}^{(T)} & N_2 = N_{\theta z}{}^{(T)} \\ S_1{}' = N_{\phi\theta}{}^{(T)} & S_1{}' = - N_{\theta\phi}{}^{(T)} \\ H_1 = M_{\phi\theta}{}^{(T)} & H_2 = - M_{\theta\phi}{}^{(T)} \\ G_1 = M_{\phi\phi}{}^{(T)} & G_2 = M_{\theta\theta}{}^{(T)} \end{array}\right\} \quad 2.1.7$$

For an explanation of the meanings of the various terms of Equations 2.1.7, see Figures 2.1.1, 2.1.2 and 2.1.3.

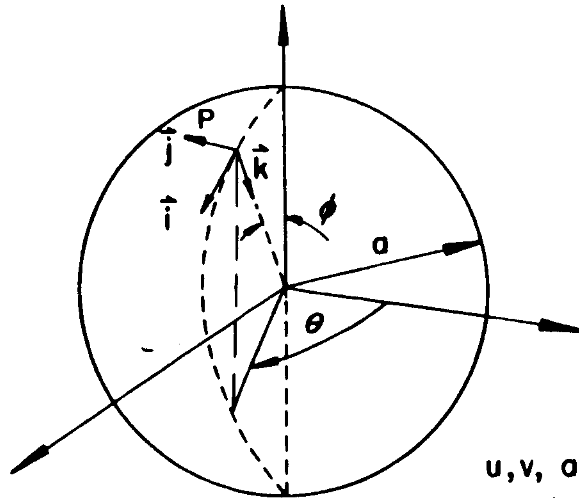
On using Equations 2.1.7 in the equations of Love, p. 523, we find:

$$p_1{}' = \frac{\operatorname{Cos} \phi}{a \operatorname{Sin}^2 \phi} \frac{\partial w^{(T)}}{\partial \theta} + \frac{1}{a \operatorname{Sin} \phi} \frac{\partial^2 w^{(T)}}{\partial\theta\, \partial \phi} \quad 2.1.8$$

$$\left.\begin{array}{l} p_2{}' = \operatorname{Sin} \phi + \frac{1}{a \operatorname{Sin} \phi} \frac{\partial^2 w^{(T)}}{\partial \theta^2} + \frac{1}{a} \frac{\partial v^{(T)}}{\partial \theta} \\ \qquad + \frac{\operatorname{Cos} \phi}{a} \frac{\partial w^{(T)}}{\partial \phi} + \frac{\operatorname{Cos} \phi}{a} u^{(T)} \end{array}\right\} \quad 2.1.9$$

$$q_1{}' = - 1 - \frac{1}{a} \frac{\partial^2 w^{(T)}}{\partial \phi^2} - \frac{1}{a} \frac{\partial u^{(T)}}{\partial \phi} \quad 2.1.10$$

$$\left.\begin{array}{l} q_2{}' = - \frac{1}{a} \frac{\partial^2 w^{(T)}}{\partial \theta\, \partial \phi} - \frac{1}{a} \frac{\partial u^{(T)}}{\partial \theta} + \frac{\operatorname{Cot} \phi}{a} \frac{\partial w^{(T)}}{\partial \theta} \\ \qquad + \frac{\operatorname{Cos} \phi}{a} v^{(T)} - \frac{\operatorname{Sin} \phi}{a} \frac{\partial v^{(T)}}{\partial \phi} \end{array}\right\} \quad 2.1.11$$



u,v, and w are the components at displacements in the directions of $\bar{i}$, $\bar{j}$ and $\bar{k}$ respectively

FIG. 2.1.1

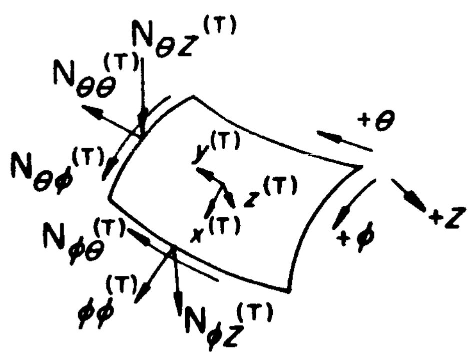


FIG. 2.1.2

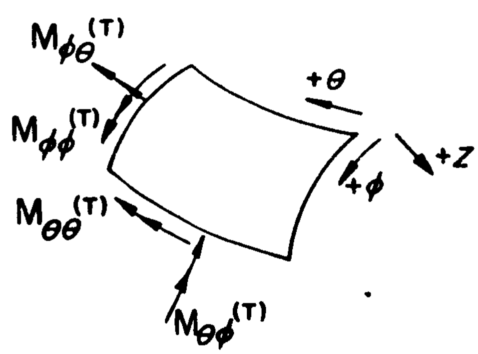


FIG. 2.1.3

$$r_1' = \frac{1}{a} \frac{\partial^2 v^{(T)}}{\partial \phi^2} + \frac{1}{a \operatorname{Sin} \phi} \frac{\partial w^{(T)}}{\partial \theta} + \frac{v^{(T)}}{a} \qquad 2.1.12$$

$$r_2' = \operatorname{Cos} \phi + \frac{1}{a} \frac{\partial^2 v^{(T)}}{\partial \theta \partial \phi} - \frac{\operatorname{Sin} \phi}{a} \frac{\partial w^{(T)}}{\partial \phi} - \frac{\operatorname{Sin} \phi}{a} u^{(T)} \qquad 2.1.13$$

For a uniformly pressurized sphere with additional loads, we note:

$$\left.\begin{array}{ll}
N_{\phi\phi}^{(T)} = N_{\phi\phi} + N & N_{\theta\theta}^{(T)} = N_{\theta\theta} + N \\
N_{\theta\phi}^{(T)} = N_{\phi\theta}^{(T)} = N_{\theta\phi} & N_{\phi z}^{(T)} = N_{\phi z} \\
N_{\theta z}^{(T)} = N_{\theta z} & M_{\phi\phi}^{(T)} = M_{\phi\phi} \\
M_{\theta\theta}^{(T)} = M_{\theta\theta} & M_{\theta\phi}^{(T)} = M_{\theta\phi} \\
u^{(T)} = u & v^{(T)} = v \\
w^{(T)} = w_o + w & x^{(T)} = x \\
y^{(T)} = y & Z^{(T)} = Z - p
\end{array}\right\} \qquad 2.1.14$$

where

$$\left.\begin{array}{l}
N = \frac{pa}{2} \\
w_o - \frac{pa(1 - \nu)}{2E}
\end{array}\right\} \qquad 2.1.15$$

On substituting Equations 2.1.7 - 2.1.15 in 2.1.1 - 2.1.6, we find

$$\left.\begin{array}{l}
\frac{\partial N_{\phi\phi}}{\partial \phi} + \frac{1}{\operatorname{Sin} \phi} \frac{\partial N_{\theta\phi}}{\partial \theta} + \operatorname{Cot} \phi\, N_{\phi\phi} - \operatorname{Cot} \phi\, N_{\theta\theta} \\
- N_{\phi z} + \left(\frac{1}{a} \frac{\partial w}{\partial \phi} + \frac{u}{a} - \frac{1}{a \operatorname{Sin} \phi} \frac{\partial^2 v}{\partial \theta \partial \phi}\right) N + aX = O
\end{array}\right\} \qquad 2.1.16$$

$$\left.\begin{aligned} &\frac{\partial N_{\theta\phi}}{\partial \phi} \quad \frac{1}{\mathrm{Sin}\ \phi} \frac{\partial N_{\theta\phi}}{\partial \theta} + 2\ \mathrm{Cot}\ \phi\ N_{\theta\phi} - N_{\theta z} \\ &\quad + \left(\frac{1}{a} \frac{\partial^2 v}{\partial \phi^2} + \frac{1}{a\ \mathrm{Sin}\ \phi} \frac{\partial w}{\partial \theta} + \frac{v}{a} \right) N + a\,Y = 0 \end{aligned}\right\} \qquad 2.1.17$$

$$\left.\begin{aligned} &\frac{\partial N_{\phi z}}{\partial \phi} + \frac{1}{\mathrm{Sin}\ \phi} \frac{\partial N_{\theta z}}{\partial \theta} + N_{\phi\phi} + N_{\theta\theta} + \mathrm{Cot}\ \phi\ N_{\phi z} \\ &\quad + \left(\frac{1}{a} \frac{\partial^2 w}{\partial \phi^2} + \frac{1}{a} \frac{\partial u}{\partial \phi} + \frac{1}{a\ \mathrm{Sin}^2\ \phi} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{a \mathrm{Sin}\ \phi} \frac{\partial v}{\partial \theta} \right. \\ &\quad \left. + \mathrm{Cot}\ \phi \frac{\partial w}{\partial \phi} + \frac{\mathrm{Cot}\ \phi}{a} u \right) N + a\,Z = 0 \end{aligned}\right\} \qquad 2.1.18$$

$$\frac{\partial M_{\theta\phi}}{\partial \phi} - \frac{1}{\mathrm{Sin}\ \phi} \frac{\partial M_{\theta\theta}}{\partial \theta} + 2\ \mathrm{Cot}\ \phi\ M_{\theta\phi} + a\,N_{\theta z} = 0 \qquad 2.1.19$$

$$\frac{\partial M_{\phi\phi}}{\partial \phi} - \frac{1}{\mathrm{Sin}\ \phi} \frac{\partial M_{\theta\phi}}{\partial \theta} + \mathrm{Cot}\phi\ (M_{\phi\phi} - M_{\theta\theta}) - a\,N_{\phi z} = 0 \qquad 2.1.20$$

$$M_{\theta\phi} = M_{\phi\theta} \qquad 2.1.21$$

2.2 Stress-Displacement Relations

The stress resultants and stress couples (see, for example, Wang, p. 340) are

$$\left.\begin{aligned}
N_{\phi\phi} &= \frac{Eh}{(1-\nu^2)} \quad (\epsilon_{10} + \nu\,\epsilon_{20}) \\
N_{\theta\theta} &= \frac{Eh}{(1-\nu^2)} \quad (\epsilon_{20} + \nu\,\epsilon_{10}) \\
N_{\phi\theta} &= N_{\theta\phi} = \frac{Eh\,\gamma_o}{2(1+\nu)} \\
M_{\phi\phi} &= -D\,(X_1 + \nu X_2) \\
M_{\theta\theta} &= -D\,(X_2 + \nu X_1) \\
M_{\theta\phi} &= M_{\phi\theta} = D\,(1-\nu)\,X_{12}
\end{aligned}\right\} \qquad 2.2.1$$

where for a sphere

$$\left.\begin{aligned}
\epsilon_{10} &= \frac{1}{a}\,\frac{\partial u^{(T)}}{\partial\phi} - \frac{w^{(T)}}{a} \\
\epsilon_{20} &= \frac{1}{a\,\mathrm{Sin}\,\phi}\,\frac{\partial \nu^{(T)}}{\partial\theta} + \frac{\mathrm{Cot}\,\phi}{a} - u^{(T)} - \frac{w^{(T)}}{a} \\
\gamma_0 &= \frac{\mathrm{Cot}\,\phi}{a}\,v^{(T)} + \frac{1}{a}\,\frac{\partial v^{(T)}}{\partial\phi} + \frac{1}{a\,\mathrm{Sin}\,\phi}\,\frac{\partial u^{(T)}}{\partial\theta} \\
X_1 &= \frac{1}{a^2}\,\frac{\partial u^{(T)}}{\partial\phi} + \frac{1}{a^2}\,\frac{\partial^2 w^{(T)}}{\partial\phi^2} \\
X_2 &= \frac{1}{a^2\,\mathrm{Sin}\,\phi}\,\frac{\partial v^{(T)}}{\partial\theta} + \frac{1}{a^2\,\mathrm{Sin}\,\phi}\,\frac{\partial^2 w^{(T)}}{\partial\theta^2} + \frac{\mathrm{Cot}\,\phi}{a^2}\,u^{(T)} \\
&\quad + \frac{\mathrm{Cot}\,\phi}{a^2}\,\frac{\partial w^{(T)}}{\partial\phi} \\
X_{12} &= \frac{1}{2}\Big[-\frac{\mathrm{Cot}\,\phi}{a^2}\,v^{(T)} + \frac{1}{a^2}\,\frac{\partial v^{(T)}}{\partial\phi} - \frac{2\,\mathrm{Cos}\,\phi}{a^2\,\mathrm{Sin}^2\phi}\,\frac{\partial w^{(T)}}{\partial\theta} \\
&\quad + \frac{2}{a^2\,\mathrm{Sin}\phi}\,\frac{\partial w^{(T)}}{\partial\theta\,\partial\phi} + \frac{1}{a^2\,\mathrm{Sin}\,\phi}\,\frac{\partial u^{(T)}}{\partial\theta}\Big]
\end{aligned}\right\} \qquad 2.2.2$$

Introducing

$$E' = \frac{Eh}{(1 - \nu^2)} \qquad 2.2.3$$

and using Equations 2.1.14 and 2.2.2 in 2.2.1 we obtain:

$$N_{\phi\phi} = E' \left\{ \frac{1}{a} \frac{\partial u}{\partial \phi} - (1+\nu)\frac{w}{a} + \frac{\nu}{a \operatorname{Sin} \phi} \frac{\partial v}{\partial \theta} + \frac{\nu \operatorname{Cot} \phi}{a} u \right\} \qquad 2.2.4$$

$$N_{\theta\theta} = E' \left\{ \frac{1}{a \operatorname{Sin} \phi} \frac{\partial v}{\partial \theta} + \frac{\operatorname{Cot}\phi}{a} u - \frac{(1+\nu)}{a} w + \frac{\nu}{a} \frac{\partial u}{\partial \phi} \right\} \qquad 2.2.5$$

$$N_{\theta\phi} = E' \frac{(1-\nu)}{2} \left\{ - \frac{\operatorname{Cot} \phi}{a} \nu + \frac{1}{a} \frac{\partial v}{\partial \phi} + \frac{1}{a \operatorname{Sin} \phi} \frac{\partial u}{\partial \theta} \right\} \qquad 2.2.6$$

$$\left.\begin{aligned} M_{\phi\phi} = - D \Big\{ & \frac{1}{a^2} \frac{\partial u}{\partial \phi} + \frac{1}{a^2} \frac{\partial^2 w}{\partial \phi^2} + \frac{\nu}{a^2 \operatorname{Sin} \phi} \frac{\partial v}{\partial \theta} \\ & + \frac{\nu}{a^2 \operatorname{Sin}^2 \phi} \frac{\partial^2 w}{\partial \theta^2} + \frac{\nu \operatorname{Cot} \phi}{a^2} u + \frac{\nu \operatorname{Cot} \phi}{a^2} \frac{\partial w}{\partial \phi} \end{aligned}\right\} \qquad 2.2.7$$

$$\left.\begin{aligned} M_{\theta\theta} = - D \Big\{ & \frac{1}{a^2 \operatorname{Sin} \phi} \frac{\partial v}{\partial \theta} + \frac{1}{a^2 \operatorname{Sin}^2 \phi} \frac{\partial^2 w}{\partial \theta^2} + \frac{\operatorname{Cot} \phi}{a^2} u \\ & + \frac{\operatorname{Cot} \phi}{a^2} \frac{\partial w}{\partial \phi} + \frac{\nu}{a^2} \frac{\partial u}{\partial \phi} + \frac{\nu}{a^2} \frac{\partial^2 w}{\partial \phi^2} \Big\} \end{aligned}\right\} \qquad 2.2.8$$

$$\left.\begin{aligned} M_{\theta\phi} = \frac{D(1-\nu)}{2} \Big\{ & - \frac{\operatorname{Cot}\phi}{a^2} v + \frac{1}{a^2} \frac{\partial v}{\partial \phi} - \frac{2 \operatorname{Cos} \phi}{a^2 \sin^2 \phi} \frac{\partial w}{\partial \theta} \\ & + \frac{2}{a^2 \operatorname{Sin} \phi} \frac{\partial^2 w}{\partial \theta \, \partial \phi} + \frac{1}{a^2 \operatorname{Sin} \phi} \frac{\partial u}{\partial \theta} \Big\} \end{aligned}\right\} \qquad 2.2.9$$

Chapter 3.

3.0 Reduction to Ordinary Differential Equations

Since the solution, numerical or analytic, of the systems of partial differential equations derived in Chapter 2 is very difficult and time consuming, it is expedient to expand the dependent variables in a Fourier series in the independent variable θ. This expansion reduces the system of partial differential equations to a system of linear ordinary differential equations which are much more tractable.

We now postulate expansions of the dependent variables in trigonometric series of the form

$$\left.\begin{aligned}
u(\phi,\theta) &= \sum_{0}^{\infty} u^{(n)}(\phi) \begin{Bmatrix}\operatorname{Cos} n\theta \\ \operatorname{Sin} n\theta\end{Bmatrix} \\
v(\phi,\theta) &= \sum_{0}^{\infty} v^{(n)}(\phi) \begin{Bmatrix}\operatorname{Sin} n\theta \\ \operatorname{Cos} n\theta\end{Bmatrix} \\
w(\phi,\theta) &= \sum_{0}^{\infty} w^{(n)}(\phi) \begin{Bmatrix}\operatorname{Cos} n\theta \\ \operatorname{Sin} n\theta\end{Bmatrix} \\
N_{\theta\theta}(\phi,\theta) &= \sum_{0}^{\infty} N_{\theta\theta}^{(n)}(\phi) \begin{Bmatrix}\operatorname{Cos} n\theta \\ \operatorname{Sin} n\theta\end{Bmatrix} \\
N_{\phi\phi}(\phi,\theta) &= \sum_{0}^{\infty} N_{\phi\phi}^{(n)}(\phi) \begin{Bmatrix}\operatorname{Cos} n\theta \\ \operatorname{Sin} n\theta\end{Bmatrix} \\
N_{\theta\phi}(\phi,\theta) &= \sum_{0}^{\infty} N_{\theta\phi}^{(n)}(\phi) \begin{Bmatrix}\operatorname{Sin} n\theta \\ \operatorname{Cos} n\theta\end{Bmatrix} \\
N_{\theta z}(\phi,\theta) &= \sum_{0}^{\infty} N_{\theta z}^{(n)}(\phi) \begin{Bmatrix}\operatorname{Sin} n\theta \\ \operatorname{Cos} n\theta\end{Bmatrix} \\
N_{\phi z}(\phi,\theta) &= \sum_{0}^{\infty} N_{\phi z}^{(n)}(\phi) \begin{Bmatrix}\operatorname{Cos} n\theta \\ \operatorname{Sin} n\theta\end{Bmatrix} \\
M_{\theta\theta}(\phi,\theta) &= \sum_{0}^{\infty} M_{\theta\theta}^{(n)}(\phi) \begin{Bmatrix}\operatorname{Cos} n\theta \\ \operatorname{Sin} n\theta\end{Bmatrix}
\end{aligned}\right\} \quad 3.0.1$$

$$
\left.
\begin{aligned}
M_{\phi\phi}(\phi,\theta) &= \sum_0^\infty M_{\phi\phi}^{(n)}(\phi) \begin{Bmatrix}\text{Cos } n\theta \\ \text{Sin } n\theta\end{Bmatrix} \\
M_{\theta\phi}(\phi,\theta) &= \sum_0^\infty M_{\theta\phi}^{(n)}(\phi) \begin{Bmatrix}\text{Sin } n\theta \\ \text{Cos } n\theta\end{Bmatrix} \\
X(\phi,\theta) &= \sum_0^\infty X^{(n)}(\phi) \begin{Bmatrix}\text{Cos } n\theta \\ \text{Sin } n\theta\end{Bmatrix} \\
Y(\phi,\theta) &= \sum_0^\infty Y^{(n)}(\phi) \begin{Bmatrix}\text{Sin } n\theta \\ \text{Cos } n\theta\end{Bmatrix} \\
Z(\phi,\theta) &= \sum_0^\infty Z^{(n)}(\phi) \begin{Bmatrix}\text{Cos } n\theta \\ \text{Sin } n\theta\end{Bmatrix}
\end{aligned}
\right\} \quad 3.0.1
$$

where the symbol $\begin{Bmatrix}\text{Cos } n\theta \\ \text{Sin } n\theta\end{Bmatrix}$ indicates either a Cosine series or a Sine series used. In the following equations certain terms will have a double sign, e.g. $\pm$; this notation indicates the "upper sign" should be used when the "upper series" is used, etc.

Substituting equations 3.0.1 in equations 2.1.16 - 2.1.20 and 2.2.4 - 2.2.9 and dividing out common terms, we obtain the following system of ordinary differential equations.

$$
\left.
\begin{aligned}
&\frac{dN_{\phi\phi}^{(n)}}{d\phi} \pm \frac{n}{\text{Sin }\phi} N_{\theta\phi}^{(n)} + \text{Cot }\phi\, N_{\phi\phi}^{(n)} - \text{Cot }\phi\, N_{\theta\theta}^{(n)} \\
&\quad - N_{\phi z}^{(n)} + \left(\frac{dw^{(n)}}{d\phi} + u^{(n)} \mp \frac{n}{\text{Sin }\phi}\frac{dv^{(n)}}{d\phi}\right)\frac{N}{a} + a\,X^{(n)} = 0
\end{aligned}
\right\} \quad 3.0.2
$$

$$
\left.
\begin{aligned}
&\frac{dN_{\theta\phi}^{(n)}}{d\phi} \mp \frac{n}{\text{Sin }\phi} N_{\theta\theta}^{(n)} + 2\,\text{Cot }\phi\, N_{\theta\phi}^{(n)} - N_{\theta z}^{(n)} \\
&\quad + \left(\frac{d^2 v^{(n)}}{d\phi^2} \mp \frac{n}{\text{Sin }\phi} w^{(n)} + v^{(n)}\right)\frac{N}{a} + a\,Y^{(n)} = 0
\end{aligned}
\right\} \quad 3.0.3
$$

$$
\left.
\begin{aligned}
&\frac{dN_{\phi z}^{(n)}}{\partial\phi} \mp \frac{n}{\text{Sin }\phi} N_{\theta z}^{(n)} + N_{\phi\phi}^{(n)} + N_{\theta\theta}^{(n)} + \text{Cot }\phi\, N_{\phi z}^{(n)} \\
&\quad + \left(\frac{d^2 w^{(n)}}{\partial\phi^2} + \frac{du^{(n)}}{\partial\phi} - \frac{n^2}{\text{Sin}^2\phi} w^{(n)} \pm \frac{n}{\text{Sin }\phi} v^{(n)}\right. \\
&\quad \left. + \text{Cot }\phi\,\frac{dw^{(n)}}{\partial\phi} + \text{Cot }\phi\, u^{(n)}\right)\frac{N}{a} + a\,Z^{(n)} = 0
\end{aligned}
\right\} \quad 3.0.4
$$

$$\frac{dM_{\theta\phi}^{(n)}}{d\phi} \mp \frac{n}{\sin\phi} M_{\theta\phi}^{(n)} + 2 \cot\phi \; M_{\theta\phi}^{(n)} + a \; N_{\theta z}^{(n)} = 0 \qquad 3.0.5$$

$$\frac{dM_{\phi\phi}^{(n)}}{d\phi} \mp \frac{n}{\sin\phi} M_{\theta\phi}^{(n)} + \cot\phi \left(M_{\phi\phi}^{(n)} - M_{\theta\theta}^{(n)} \right) - a \, N_{\phi z}^{(n)} = 0 \qquad 3.0.6$$

$$N_{\phi\phi}^{(n)} = E' \left\{ \frac{1}{a} \frac{du^{(n)}}{d\phi} - \left(\frac{1+\nu}{a} \right) w^{(n)} \pm \frac{\nu \, n}{a \sin\phi} v^{(n)} + \frac{\nu \cot\phi}{a} u^{(n)} \right\} \qquad 3.0.7$$

$$N_{\theta\theta}^{(n)} = E' \left\{ \pm \frac{n}{a \sin\phi} v^{(n)} + \frac{\cot\phi}{a} u^{(n)} - \frac{(1+\nu)}{a} w^{(n)} + \frac{\nu}{a} \frac{du^{(n)}}{d\phi} \right\} \qquad 3.0.8$$

$$N_{\theta\phi}^{(n)} = \frac{E' (1-\nu)}{2} \left\{ - \frac{\cot\phi}{a} v^{(n)} + \frac{1}{a} \frac{dv^{(n)}}{d\phi} \mp \frac{n}{a \sin\phi} u^{(n)} \right\} \qquad 3.0.9$$

$$\left. \begin{aligned} M_{\phi\phi}^{(n)} = - D \Big\{ & \frac{1}{a^2} \frac{du^{(n)}}{d\phi} + \frac{1}{a^2} \frac{d^2 w^{(n)}}{d\phi^2} \mp \frac{\nu \, n}{a^2 \sin\phi} v^{(n)} \\ & - \frac{\nu \, n^2}{a^2 \sin^2\phi} w^{(n)} + \nu \frac{\cot\phi}{a^2} u^{(n)} + \frac{\nu \cot\phi}{a^2} \frac{dw^{(n)}}{d\phi} \Big\} \end{aligned} \right\} \qquad 3.0.10$$

$$\left. \begin{aligned} M_{\theta\theta}^{(n)} = - D \Big\{ & \pm \frac{n}{a^2 \sin\phi} v^{(n)} - \frac{n^2}{a^2 \sin^2\phi} w^{(n)} + \frac{\cot\phi}{a^2} u^{(n)} \\ & + \frac{\cot\phi}{a^2} \frac{dw^{(n)}}{d\phi} + \frac{\nu}{a^2} \frac{du^{(n)}}{d\phi} + \frac{\nu}{a^2} \frac{d^2 w^{(n)}}{d\phi^2} \Big\} \end{aligned} \right\} \qquad 3.0.11$$

$$\left. \begin{aligned} M_{\theta\phi} = D (1-\nu) \Big\{ & \mp \frac{n}{a^2 \sin\phi} \frac{dw^{(n)}}{d\phi} \pm \frac{n \cos\phi}{a^2 \sin^3\phi} w^{(n)} \\ & - \frac{\cot\phi}{a^2} \nu^{(n)} + \frac{1}{2 a^2} \frac{d\nu^{(n)}}{d\phi} \mp \frac{n}{2 a^2 \sin\phi} u^{(n)} \Big\} \end{aligned} \right\} \qquad 3.0.12$$

Chapter 4.

4.0 Reduction to First Order Form

In this chapter, the ordinary differential equations, 3.0.2 through 3.0.12, are reduced to a form particularly suited to numerical integration. The reader primarily interested in the final form of the equations is directed to Chapter 1, equations 1.1.1 through 1.1.16, where the final forms are given.

The key to successful manipulation of these equations lies in the introduction of three new dependent variables, $\lambda^{(n)}$, $\psi_{\theta}{}^{(n)}$ and $\psi_{\phi}{}^{(n)}$. The quantity $\lambda^{(n)}$ represents the slope of the median surface in the direction of the generator. $\psi_{\phi}{}^{(n)}$ is a quantity reminiscent of and related to the classical Kirchhoff shear of plate theory. The form of the quantity $\psi_{\theta}{}^{(n)}$ is suggested by the equations and is related to the effective membrane edge shear for shells of revolution. Throughout this chapter, the main purpose will be to manipulate the basic equations into a form such that on the left side of the equation we have a single first derivative while on the right side we have only the quantities themselves. This particular form is conveniently integrated by the Runge-Kutta method or some similar integration scheme.

We introduce the new dependent variable

$$\lambda^{(n)} = \frac{1}{a} \frac{dw^{(n)}}{d\phi} \qquad 4.0.1$$

Eliminating $N_{\theta z}{}^{(n)}$ between 3.2.3 and 3.2.5, we get

$$\left.\begin{aligned}&\frac{dN_{\theta\phi}^{(n)}}{d\phi} + \frac{1}{a}\,\frac{dM_{\theta\phi}^{(n)}}{d\phi} + 2\mathrm{Cot}\phi\,(N_{\theta\phi}^{(n)} + \frac{1}{a}\,M_{\theta\phi}^{(n)})\\ &\mp \frac{n}{\mathrm{Sin}\phi}\,N_{\theta\theta}^{(n)} \pm \frac{n}{a\mathrm{Sin}\phi}\,M_{\theta\theta} + [\,\frac{d^2v^{(n)}}{d\phi^2} \mp \frac{n}{\mathrm{Sin}\phi}\,w^{(n)}\\ &+ v^{(n)}]\,\frac{N}{a} + a\,Y^{(n)} = 0\end{aligned}\right\}\qquad 4.0.2$$

Eliminating $N_{\theta z}^{(n)}$ between 3.0.4 and 3.0.5, we obtain

$$\left.\begin{aligned}&\frac{dN_{\phi z}^{(n)}}{d\phi} \mp \frac{n}{a\mathrm{Sin}\phi}\,\frac{dM_{\theta\phi}^{(n)}}{d\phi} + \mathrm{Cot}\,\phi\;N_{\phi z}^{(n)} + N_{\phi\phi}^{(n)} + N_{\theta\phi}^{(n)}\\ &+ N\,\frac{d\lambda^{(n)}}{d\phi} + \frac{N}{a}\,\frac{du^{(n)}}{d\phi} - \frac{n^2N}{a\mathrm{Sin}^2\phi}\,w^{(n)} \pm \frac{nN}{a\mathrm{Sin}\phi}\,v^{(n)}\\ &+ N\,\mathrm{Cot}\,\phi\;\lambda^{(n)} + \frac{N}{a}\,\mathrm{Cot}_{\phi}\,u^{(n)} - \frac{n^2}{a\mathrm{Sin}^2\phi}\,M_{\theta\phi}^{(n)}\\ &\mp \frac{2n\mathrm{Cot}\,\phi}{a\mathrm{Sin}\phi}\,M_{\theta\phi}^{(n)} + aZ^{(n)}\end{aligned}\right\}\qquad 4.0.3$$

We now introduce a second new dependent variable,

$$\psi_{\phi}^{(n)} = a\mathrm{Sin}\,\phi\;N_{\phi z}^{(n)} \mp nM_{\theta\phi}^{(n)} \qquad 4.0.4$$

which implies

$$\frac{d\psi_{\phi}^{(n)}}{d\phi} = a\mathrm{Sin}\phi\;d\,\frac{N_{\phi z}^{(n)}}{d\phi} + a\mathrm{Cos}_{\phi}\,N_{\phi z} \mp n\,\frac{dM_{\theta\phi}}{d\phi} \qquad 4.0.5$$

Substituting 4.0.5 in 4.0.3 we obtain

$$\left.\begin{aligned}
&\frac{1}{a\,\text{Sin}\,\phi}\;\frac{d\psi_{\phi}^{(n)}}{d\varphi} + N_{\phi\phi}^{(n)} + N_{\theta\theta}^{(n)} + N\,\frac{d\lambda^{(n)}}{d\,\phi} + \frac{N}{a}\;\frac{du^{(n)}}{d\phi}\\
&- \frac{n^2 N}{a\,\text{Sin}^2\phi}\;w^{(n)} \pm \frac{nN}{a\text{Sin}\phi}\;v^{(n)} + N\,\text{Cot}\,\phi\;\lambda^{(n)}\\
&+ \frac{N}{a}\;\text{Cot}\,\phi\;u^{(n)} - \frac{n^2}{a\text{Sin}^2\phi}\;M_{\theta\phi}^{(n)} \mp \frac{2n\text{Cot}\phi}{a\text{Sin}\phi}\;M_{\theta\phi}^{(n)}\\
&+ a\;Z^{(n)} = 0
\end{aligned}\right\} \quad 4.0.6$$

Introducing the third new dependent variable

$$\left.\psi_{\theta}^{(n)} = a\,\text{Sin}\,\phi\;\;N_{\theta\phi}^{(n)} + \text{Sin}\phi\;\;M_{\theta\phi}^{(n)}\right\} \quad 4.0.7$$

we also get

$$\left.\begin{aligned}
\frac{d\psi_{\theta}^{(n)}}{d\phi} = {} & a\text{Sin}\phi\;\;\frac{dN_{\theta\phi}^{(n)}}{d\phi} + a\,\text{Cos}\phi\;\;N_{\theta\phi}^{(n)} + \text{Cos}\phi\;\;M_{\theta\phi}^{(n)}\\
& + \text{Sin}\,\phi\;\;\frac{dM_{\theta\phi}^{(n)}}{d\phi}
\end{aligned}\right\} \quad 4.0.8$$

Substituting 4.0.7 and 4.0.8 in 4.0.2 we obtain:

$$\left.\begin{aligned}
\frac{d\psi_{\theta}^{(n)}}{d\phi} = {} & -\;\{N\,\text{Sin}\,\phi\}\;\;v^{(n)} \pm \{n\,N\}\;\;w^{(n)} \pm \{na\}\;\;N_{\theta\theta}^{(n)}\\
& \mp \{\,n\,\}\;\;M_{\theta\theta}^{(n)} - \{\,\text{Cot}\phi\,\}\;\psi_{\theta}^{(n)} - \{\,N\text{Sin}\,\phi\,\}\,\frac{d^2v^{(n)}}{d\phi^2}\\
& - \{\,a^2\,\text{Sin}\,\phi\,\}\;\;Y^{(n)}
\end{aligned}\right\} \quad 4.0.9$$

From 3.0.7

$$\left.\begin{aligned}\frac{du^{(n)}}{d\phi} &= -\{\nu\operatorname{Cot}\phi\}u^{(n)} \mp \left\{\frac{\nu n}{\sin\phi}\right\}v^{(n)}\\ &+\{(1+\nu)\}w^{(n)} + \left\{\frac{a}{E'}\right\}N_{\phi\phi}^{(n)}\end{aligned}\right\} \quad 4.0.10$$

Eliminating $\frac{du^{(n)}}{du}$ between Equations 3.0.7 and 3.0.8, we obtain

$$\left.\begin{aligned}N_{\theta\phi}^{(n)} &= \left\{\frac{(1-\nu^2)E'\operatorname{Cot}\phi}{a}\right\}u^{(n)} \pm \left\{\frac{n(1-\nu^2)E'}{a\operatorname{Sin}\phi}\right\}v^{(n)} - \left\{\frac{(1-\nu^2)E'}{a}\right\}w^{(n)}\\ &+\{\nu\}N_{\phi\phi}\end{aligned}\right\} \quad 4.0.11$$

Eliminating $\left[\frac{1}{a^2}\frac{du^{(n)}}{d\varphi} + \frac{1}{a}\frac{d\lambda^{(n)}}{d\phi}\right]$ between 3.0.10 and 3.0.11, we obtain

$$\left.\begin{aligned}M_{\theta\theta}^{(n)} &= -\left\{\frac{(1-\nu^2)\,D\operatorname{Cot}\phi}{a^2}\right\}u^{(n)} \mp \left\{\frac{n(1-\nu^2)D}{a^2\operatorname{Sin}\phi}\right\}v^{(n)}\\ &+\left\{\frac{n^2(1-\nu^2)D}{a^2\operatorname{Sin}^2\phi}\right\}w^{(n)} - \left\{\frac{(1-\nu^2)\,D\operatorname{Cot}\phi}{a}\right\}\lambda^{(n)}\\ &+\{\nu\}M_{\phi\phi}^{(n)}\end{aligned}\right\} \quad 4.0.12$$

Substitution of 4.0.11 and 4.0.12 into 4.0.9 yields

$$\left.\begin{aligned}\frac{d\psi_\theta^{(n)}}{d\phi} &= -\{N\operatorname{Sin}\phi\}\frac{d^2v^{(n)}}{d\phi^2} \pm \left\{n(1-\nu^2)E'\operatorname{Cot}\phi\left[1+\frac{D}{a^2E'}\right]\right\}u^{(n)}\\ &+\left\{-N\operatorname{Sin}\phi + \frac{n^2(1-\nu^2)E'}{\operatorname{Sin}\phi}\left[1+\frac{D}{a^2E'}\right]\right\}v^{(n)}\\ &\mp\left\{-nN + n(1-\nu^2)E'\left[1+\frac{n^2D}{a^2E'\operatorname{Sin}^2\phi}\right]\right\}w^{(n)} \pm \left\{\frac{n(1-\nu^2)D\operatorname{Cot}\phi}{a}\right\}\lambda^{(n)}\\ &\pm\{\nu na\}N_{\phi\phi}^{(n)} \mp \{\nu n\}M_{\phi\phi}^{(n)} - \{\operatorname{Cot}\phi\}\psi_\theta^{(n)} - \{a^2\operatorname{Sin}\phi\}Y^{(n)}\end{aligned}\right\} \quad 4.0.13$$

and substitution of 4.0.7 into 3.0.9 yields

$$\left.\begin{aligned}\frac{dv^{(n)}}{d\phi} &= \pm\left\{\frac{n}{\operatorname{Sin}\phi}\right\}u^{(n)} + \{\operatorname{Cot}\phi\}\, v^{(n)} \\ &+ \left\{\frac{2}{E'(1-\nu)\operatorname{Sin}\phi}\right\}\psi_\theta^{(n)} - \left\{\frac{2}{E'(1-\nu)}\right\} M_{\theta\phi}^{(n)}\end{aligned}\right\} \quad 4.0.14$$

Substitution of 3.2.12 and 4.0.1 into 4.0.14 leads to

$$\left.\begin{aligned}\left\{1+\frac{D}{a^2E'}\right\}\frac{dv^{(n)}}{d\phi} &= \pm\left\{\frac{n}{\operatorname{Sin}\phi}\left[1+\frac{D}{a^2E'}\right]\right\}u^{(n)} \\ &+ \left\{\operatorname{Cot}\phi\left[1+\frac{D}{a^2E'}\right]\right\}v^{(n)} \mp \left\{\frac{2nD\operatorname{Cot}\phi}{a^2E'\operatorname{Sin}\phi}\right\} w^{(n)} \\ &\pm\left\{\frac{2nD}{aE'\operatorname{Sin}\phi}\right\}\lambda^{(n)} + \left\{\frac{2}{E'(1-\nu)\operatorname{Sin}\phi}\right\}\psi_\theta^{(n)}\end{aligned}\right\} \quad 4.0.15$$

Substituting 4.0.10 into 3.0.10, we get

$$\left.\begin{aligned}\frac{d\lambda^{(n)}}{d\phi} &= \left\{\frac{\nu n^2}{a\operatorname{Sin}^2\phi} - \frac{(1+\nu)}{a}\right\} w^{(n)} - \{\nu\operatorname{Cot}\psi\}\lambda^{(n)} \\ &- \left\{\frac{1}{E'}\right\} N_{\phi\phi}^{(n)} - \left\{\frac{a}{D}\right\} M_{\phi\phi}^{(n)}\end{aligned}\right\} \quad 4.0.16$$

Substituting 4.0.14 into 3.0.12, we obtain

$$\left.\begin{aligned}M_{\theta\phi}^{(n)} &= \pm\left\{\frac{nD(1-\nu)\operatorname{Cot}\phi}{a^2\operatorname{Sin}\phi\left(1+\frac{D}{a^2E'}\right)}\right\} w^{(n)} \\ &\mp\left\{\frac{nD(1-\nu)}{a\operatorname{Sin}\phi\left(1+\frac{D}{a^2E'}\right)}\right\}\lambda^{(n)} + \left\{\frac{D}{a^2E'\operatorname{Sin}\phi\left(1+\frac{D}{a^2E'}\right)}\right\}\psi_\theta^{(n)}\end{aligned}\right\} \quad 4.0.17$$

Substitution of 4.0.4, 4.0.12 and 4.0.17 into 3.0.6 leads to

$$\left.\begin{aligned}\frac{dM_{\phi\phi}^{(n)}}{d\phi} = &- \left\{ \frac{(1-\nu^2)D\,Cot^2\phi}{a^2} \right\} u^{(n)} \mp \left\{ \frac{n(1-\nu^2)\,D\,Cot\,\phi}{a^2\,Sin\phi} \right\} v^{(n)} \\ &+ \left\{ \frac{n^2 D(1-\nu)\,Cot\phi}{a^2 Sin^2\phi} \left[\frac{2}{(1+\frac{D}{a^2 E'})} + (1+\nu) \right] \right\} w^{(n)} \\ &- \left\{ \frac{(1-\nu)D}{a} \left[\frac{2n^2}{Sin^2\phi(1+\frac{D}{a^2E'})} + (1+\nu)Cot^2\phi \right] \right\} \lambda^{(n)} \\ &- \{ (1-\nu)\,Cot\,\phi \}\, M_{\phi\phi}^{(n)} + \left\{ \frac{1}{Sin\phi} \right\} \psi_\phi^{(n)} \\ &\pm \left\{ \frac{2\,n\,D}{a^2 E' Sin^2\phi\,(1+\frac{D}{a^2E'})} \right\} \psi_\theta^{(n)} \end{aligned}\right\} \quad 4.0.18$$

Upon substituting 4.0.17 into 4.0.4, we obtain

$$\left.\begin{aligned}N_{\phi z}^{(n)} = &\left\{ \frac{n^2 D(1-\nu)\,Cot\phi}{a^3 Sin^2\phi\,(1+\frac{D}{a^2E'})} \right\} w^{(n)} - \left\{ \frac{n^2 D(1-\nu)}{a^2 Sin^2\phi\,(1+\frac{D}{a^2E'})} \right\} \lambda^{(n)} \\ &\pm \left\{ \frac{n\,D}{a^2 E' Sin^2\phi\,(1+\frac{D}{a^2E'})} \right\} \psi_\theta^{(n)} + \left\{ \frac{1}{a Sin\phi} \right\} \psi_\phi^{(n)} \end{aligned}\right\} \quad 4.0.19$$

Substituting 4.0.14 into 4.0.7, we get

$$\left.\begin{aligned}N_{\theta\phi}^{(n)} = &\mp\left\{ \frac{nD(1-\nu)\,Cot\phi}{a^3 Sin\phi(1+\frac{D}{a^2E'})} \right\} w^{(n)} \pm \left\{ \frac{nD(1-\nu)}{a^2\,Sin\phi\,(1+\frac{D}{a^2E'})} \right\} \lambda^{(n)} \\ &+ \left\{ \frac{1}{a Sin\phi\,(1+\frac{D}{a^2E'})} \right\} \psi_\theta^{(n)} \end{aligned}\right\} \quad 4.0.20$$

We now substitute 4.0.11, 4.0.15, 4.0.19 and 4.0.20 into 3.0.2, obtaining

$$
\begin{aligned}
\frac{dN_{\phi\phi}^{(n)}}{d\phi} = & \left\{\frac{E'}{a}\left[(1-\nu^2)\operatorname{Cot}^2\phi - \frac{N}{E'}\left(1-\frac{n^2}{\operatorname{Sin}^2\phi}\right)\right]\right\} u^{(n)} \\
& \pm \left\{\frac{nE'\operatorname{Cot}\phi}{a\operatorname{Sin}\phi}\left[(1-\nu^2)+\frac{N}{E'}\right]\right\} v^{(n)} \\
& + \left\{-\frac{(1-\nu^2)E'\operatorname{Cot}\phi}{a} + \frac{2n^2 D\operatorname{Cot}\phi}{a^2\operatorname{Sin}^2\phi\left(1+\frac{D}{a^2E'}\right)}\left[(1-\nu)-\frac{N}{E'}\right]\right\} w^{(n)} \\
& - \left\{N + \frac{2n^2D}{a^2\operatorname{Sin}^2\left(1+\frac{D}{a^2E'}\right)}\left[(1-\nu)-\frac{N}{E'}\right]\right\}\lambda^{(n)} - \{(1-\nu)\operatorname{Cot}\phi\}N_{\phi\phi}^{(n)} \\
& + \left\{\frac{1}{a\operatorname{Sin}\phi}\right\}\psi_\phi^{(n)} \pm \left\{\frac{n}{a\operatorname{Sin}^2\phi}\left[1+\frac{2N}{E'(1-\nu)\left(1+\frac{D}{a^2E'}\right)}\right]\right\}\psi_\theta^{(n)} \\
& - \{a\}X^{(n)}
\end{aligned}
\qquad 4.0.21
$$

Substitution of 4.0.10, 4.0.11, 4.0.12, 4.0.16, and 4.0.17 into 4.0.6 leads to

$$
\begin{aligned}
\frac{d\psi_\phi^{(n)}}{d\phi} = &- \left\{ (1-\nu) E' \operatorname{Cos}\phi \left[(1+\nu) + \frac{N}{E'} + \frac{n^2 D (1+\nu)}{a^2 E' \operatorname{Sin}^2\phi} \right] \right\} u^{(n)} \\
&\mp \left\{ (1-\nu) E' n \left[(1+\nu) + \frac{N}{E'} + \frac{n^2 (1+\nu) D}{a^2 E' \operatorname{Sin}^2\phi} \right] \right\} v^{(n)} \\
&+ \left\{ \frac{(1-\nu^2) E'}{\operatorname{Sin}\phi} \left[\operatorname{Sin}^2\phi + \frac{n^2 N}{E'(1+\nu)} + \frac{n^4 D}{a^2 E' \operatorname{Sin}^2\phi} + \frac{2n^2 D \operatorname{Cot}^2\phi}{a^2 E' (1 + \frac{D}{a^2 E'})(1+\nu)} \right] \right\} w^{(n)} \\
&- \left\{ (1-\nu) a E' \operatorname{Cot}\phi \left[\frac{n^2 (1+\nu) D}{a^2 E' \operatorname{Sin}\phi} + \frac{2n^2 D}{a^2 E' \operatorname{Sin}\phi (1 + \frac{D}{a^2 E'})} + \frac{N}{E'} \operatorname{Sin}\phi \right] \right\} \lambda^{(n)} \\
&- \left\{ (1+\nu) a \operatorname{Sin}\phi \right\} N_{\phi\phi}^{(n)} + \left\{ \frac{N}{D} a^2 \operatorname{Sin}\phi + \frac{\nu n^2}{\operatorname{Sin}\phi} \right\} M_{\phi\phi}^{(n)} \\
&\pm \left\{ \frac{2nD \operatorname{Cot}\phi}{a^2 E' \operatorname{Sin}\phi (1 + \frac{D}{a^2 E'})} \right\} \psi_\theta^{(n)} - \left\{ a^2 \operatorname{Sin}\phi \right\} Z^{(n)}
\end{aligned}
\qquad 4.0.22
$$

If we presume that D, E' and ν are constants, upon differentiating Equation 4.0.15, we obtain

$$\{1+\frac{D}{a^2E'}\}\frac{d^2v^{(n)}}{d\phi^2} = \pm\{\frac{-n\,Cot\phi}{Sin\,\phi}[\,1+\frac{D}{a^2E'}\,]\}\,u^{(n)}$$

$$\pm\{\frac{n}{Sin\,\phi}[\,1+\frac{D}{a^2E'}\,]\}\frac{du^{(n)}}{d\phi} - \{\frac{[\,1+\frac{D}{a^2E'}\,]}{sin^2\phi}\}\,v^{(n)}$$

$$+\{Cot\,\phi\,[\,1+\frac{D}{a^2E'}\,]\}\frac{dv^{(n)}}{d\phi} \pm\{\frac{2nD}{a^2E'}[\frac{Cos^2\phi+1}{Sin^3\phi}]\}\,w^{(n)}$$

$$\mp\{\frac{4\,n\,D\,Cot\,\phi}{a\,E'\,Sin\,\phi}\}\lambda^{(n)} \pm\{\frac{2\,n\,D}{a\,E'\,Sin\,\phi}\}\frac{d\lambda^{(n)}}{d\phi}$$

$$-\{\frac{2\,Cot\,\phi}{E'(1-\nu)\,Sin\,\phi}\}\psi_\theta^{(n)} + \{\frac{2}{E'(1-\nu)\,Sin\phi}\}\frac{d\psi_\theta^{(n)}}{d\phi} \qquad 4.0.23$$

Now, substituting 4.0.10, 4.0.15 and 4.0.16 into 4.0.23, we get

$$\{1+\frac{D}{a^2E'}\}\frac{d^2v}{d\phi^2} = \mp\{\frac{\nu n\,Cot\,\phi}{Sin\phi}[\,1+\frac{D}{a^2E'}\,]\}\,u^{(n)}$$

$$-\{[\frac{\nu n^2}{Sin^2\phi}+1][\,1+\frac{D}{a^2E'}\,]\}\,v^{(n)}$$

$$\pm\{\frac{n(1+\nu)}{Sin\,\phi}[\,1+\frac{D}{a^2E'}\,] + \frac{2\,n\,D}{a^2E'Sin^3\phi} + \frac{2\,\nu n^3D}{a^2E'Sin^3\phi}$$

$$-\frac{2\,n\,(1+\nu)\,D}{a^2\,E'\,Sin\,\phi}\}\,w^{(n)} \mp \{\frac{2\,n\,D\,(1+\nu)\,Cot}{a^2\,E'\,Sin\,\phi}\}\,\lambda^{(n)}$$

$$\pm\{\frac{n\,a}{E'\,Sin\,\phi}[\,1+\frac{D}{a^2E'} - \frac{2\,n\,D}{a(E')^2\,Sin\,\phi}\}\,N_{\phi\phi}^{(n)} \mp \{\frac{2\,n}{E'Sin\phi}\}\,M_{\phi\phi}^{(n)}$$

$$+\{\frac{2}{E'(1-\nu)\,Sin\,\phi}\}\frac{d\,\psi_\theta^{(n)}}{d_\phi} \qquad 4.0.24$$

Substitution of 4.0.24 into 4.0.13 yields

$$
\begin{aligned}
&\left\{1 + \frac{2N}{E'(1-\nu)\left(1+\frac{D}{a^2E'}\right)}\right\} \frac{d\psi_\theta}{d\phi} = \pm \left\{nE'\operatorname{Cot}\phi\left[\frac{\nu N}{E'}\right.\right. \\
&\left.\left. + (1-\nu^2)\left(1+\frac{D}{a^2E'}\right)\right]\right\} u^{(n)} + \left\{\frac{n^2E'}{\operatorname{Sin}\phi}\left[\nu \frac{N}{E'}\right.\right. \\
&\left.\left. + (1-\nu^2)\left(1+\frac{D}{a^2E'}\right)\right]\right\} v^{(n)} \mp \left\{nE'\left[\nu \frac{N}{E'}\right.\right. \\
&\left.\left. + \frac{2N}{E'} \frac{D(1+\nu n^2 - (1+\nu)\operatorname{Sin}\phi)}{a^2E'\operatorname{Sin}^2\phi\left(1+\frac{D}{a^2E'}\right)} + (1-\nu^2)\left(1+\frac{n^2D}{a^2E'\operatorname{Sin}^2\phi}\right)\right]\right\} w^{(n)} \\
&\pm \left\{\frac{nD(1+\nu)\operatorname{Cot}}{a}\left[\frac{2N}{E'\left(1+\frac{D}{a^2E'}\right)} + (1-\nu)\right]\right\}\lambda^{(n)} \\
&\pm \left\{na\left[\nu - \frac{N}{E'} + \frac{2N}{E'} \frac{D}{a^2E'\left(1+\frac{D}{a^2E'}\right)}\right]\right\} N_{\phi\phi}^{(n)} \\
&\pm \left\{n\left[\frac{2N}{E'\left(1+\frac{D}{a^2E'}\right)} - \nu\right]\right\} M_{\phi\phi}^{(n)} - \{\operatorname{Cot}\phi\}\psi_\theta^{(n)} \\
&- \{a^2\operatorname{Sin}\phi\}\, Y^{(n)}
\end{aligned}
\qquad 4.0.25
$$

From 3. 0. 5, we have

$$
N_{\theta z}^{(n)} = -\frac{1}{a}\frac{dM_{\theta\phi}^{(n)}}{d\phi} \mp \frac{n}{a\operatorname{Sin}\phi} M_{\theta\theta}^{(n)} - \frac{2\operatorname{Cot}\phi}{a} M_{\theta\phi}^{(n)} \qquad 4.0.26
$$

Upon differentiating 4. 0. 17, one gets

$$\left.\begin{aligned}
\frac{dM_{\theta\phi}}{d\phi} = & \mp \left\{ \frac{nD(1-\nu)\,(\mathrm{Cos}^2\phi+1)}{a^2\,\mathrm{Sin}^3\phi\,\left(1+\frac{D}{a^2E'}\right)} \right\} w^{(n)} \\
& \pm \left\{ \frac{2nD(1-\nu)\,\mathrm{Cot}\phi}{a\,\mathrm{Sin}\,\phi\,\left(1+\frac{D}{a^2E'}\right)} \right\} \lambda^{(n)} \\
& \mp \left\{ \frac{nD(1-\nu)}{a\,\mathrm{Sin}\,\phi\,\left(1+\frac{D}{a^2E'}\right)} \right\} \frac{d\lambda^{(n)}}{d\phi} - \left\{ \frac{D\,\mathrm{Cot}\,\phi}{a^2E'\,\mathrm{Sin}\,\phi\,\left(1+\frac{D}{a^2E'}\right)} \right\} \psi_\theta^{(n)} \\
& + \left\{ \frac{D}{a^2E'\,\mathrm{Sin}\,\phi\left(1+\frac{D}{a^2E'}\right)} \right\} \frac{d\psi_\theta^{(n)}}{d\phi}
\end{aligned}\right\} \quad 4.0.27$$

Substitution of 4.0.27 into 4.0.26 yields

$$\left.\begin{aligned}
N_{\theta z}^{(n)} = & \pm \left\{ \frac{nD(1-\nu)\,(\mathrm{Cos}^2\phi+1)}{a^3\,\mathrm{Sin}^3\phi\,\left(1+\frac{D}{a^2E'}\right)} \right\} w^{(n)} \\
& \mp \left\{ \frac{2nD(1-\nu)\,\mathrm{Cot}\phi}{a^2\,\mathrm{Sin}\,\phi\,\left(1+\frac{D}{a^2E'}\right)} \right\} \lambda^{(n)} \pm \left\{ \frac{nD(1-\nu)}{a^2\,\mathrm{Sin}\phi\,\left(1+\frac{D}{a^2E'}\right)} \right\} \frac{d\lambda^{(n)}}{d\phi} \\
& + \left\{ \frac{D\,\mathrm{Cot}\,\phi}{a^3E'\,\mathrm{Sin}\phi\left(1+\frac{D}{a^2E'}\right)} \right\} \psi_\theta^{(n)} - \left\{ \frac{D}{a^3E'\,\mathrm{Sin}\,\phi\left(1+\frac{D}{a^2E'}\right)} \right\} \frac{d\psi_\theta^{(n)}}{d\phi} \\
& \mp \left\{ \frac{n}{a\,\mathrm{Sin}\,\phi} \right\} M_{\theta\theta}^{(n)} - \left\{ \frac{2\,\mathrm{Cot}\phi}{a} \right\} M_{\theta\phi}^{(n)}
\end{aligned}\right\} \quad 4.0.28$$

4.1 Summary of First Order Equations

From the previous section we summarize the results as follows:

$$\left.\begin{aligned}\frac{du^{(n)}}{d\phi} &= -\{\nu \operatorname{Cot}\phi\}\, u^{(n)} \mp \left\{\frac{\nu n}{\operatorname{Sin}\phi}\right\} v^{(n)} + \{(1+\nu)\}\, w^{(n)} \\ &\quad + \left\{\frac{a}{E'}\right\} N_{\phi\phi}^{(n)}\end{aligned}\right\} \quad 4.1.1$$

$$\left.\begin{aligned}\frac{dv^{(n)}}{d\phi} &= \pm \left\{\frac{n}{\operatorname{Sin}\phi}\right\} u^{(n)} + \{\operatorname{Cot}\phi\}\, v^{(n)} \\ &\quad \mp \left\{\frac{2\, n D \operatorname{Cot}\phi}{a^2 E' \operatorname{Sin}\phi \left(1+\frac{D}{a^2E'}\right)}\right\} w^{(n)} \pm \left\{\frac{2 n D}{a E' \operatorname{Sin}\phi \left(1+\frac{D}{a^2E'}\right)}\right\}\lambda^{(n)} \\ &\quad + \left\{\frac{2}{E'(1-\nu)\ \operatorname{Sin}\phi\ \left(1+\frac{D}{a^2E'}\right)}\right\}\psi_\theta^{(n)}\end{aligned}\right\} \quad 4.1.2$$

$$\left.\frac{dw^{(n)}}{d\phi} = a\ \lambda^{(n)}\right\} \quad 4.1.3$$

$$\left.\begin{aligned}\frac{d\lambda}{d\phi} &= \left\{\frac{\nu n^2}{a \operatorname{Sin}^2\phi} - \frac{(1+\nu)}{a}\right\} w^{(n)} - \{\nu \operatorname{Cot}\phi\}\, \lambda^{(n)} \\ &\quad - \left\{\frac{1}{E'}\right\} N_{\phi\phi}^{(n)} - \left\{\frac{a}{D}\right\} M_{\phi\phi}^{(n)}\end{aligned}\right\} \quad 4.1.4$$

$$\left.\begin{aligned}
\frac{dN_{\phi\phi}^{(n)}}{d\phi} = {} & \left\{\frac{E'}{a}\left[(1-\nu^2)\,\mathrm{Cot}^2\phi - \frac{N}{E'}\left(1-\frac{n^2}{\mathrm{Sin}^2\phi}\right)\right]\right\} u^{(n)} \\
& \pm \left\{\frac{nE'\,\mathrm{Cot}\phi}{a\,\mathrm{Sin}\,\phi}\left[(1-\nu^2)+\frac{N}{E'}\right]\right\} v^{(n)} \\
& + \left\{-\frac{(1-\nu^2)\,E'\,\mathrm{Cot}\phi}{a} + \frac{2n^2\,D\,\mathrm{Cot}}{a^3\,\mathrm{Sin}^2\phi\left(1+\frac{D}{a^2E'}\right)}\left[(1-\nu)-\frac{N}{E'}\right]\right\} w^{(n)} \\
& - \left\{N + \frac{2n^2 D\left[(1-\nu)-\frac{N}{E'}\right]}{a^2\,\mathrm{Sin}^2\phi\left(1+\frac{D}{a^2E'}\right)}\right\}\lambda^{(n)} - \{(1-\nu)\,\mathrm{Cot}\phi\}\,N_{\phi\phi}^{(n)} \\
& + \left\{\frac{1}{a\,\mathrm{Sin}\,\phi}\right\}\psi_\phi^{(n)} \pm \left\{\frac{n}{a\,\mathrm{Sin}^2\phi}\left[1+\frac{2N}{E'(1-\nu)\left(1+\frac{D}{a^2E'}\right)}\right]\right\}\psi_\theta^{(n)} \\
& - \{a\}\,X^{(n)}
\end{aligned}\right\} \quad 4.1.5$$

$$\left.\begin{aligned}
\frac{dM_{\phi\phi}^{(n)}}{d\phi} = {} & -\left\{\frac{(1-\nu^2)\,D\,\mathrm{Cot}^2\phi}{a^2}\right\} u^{(n)} \mp \left\{\frac{n(1-\nu^2)\,D\,\mathrm{Cot}\,\phi}{a^2\,\mathrm{Sin}\,\phi}\right\} v^{(n)} \\
& + \left\{\frac{n^2 D(1-\nu)\,\mathrm{Cot}\phi}{a^2\,\mathrm{Sin}^2\phi}\left[\frac{2}{\left(1+\frac{D}{a^2E'}\right)} + (1+\nu)\right]\right\} w^{(n)} \\
& - \left\{\frac{(1-\nu)\,D}{a}\left[\frac{2n^2}{\mathrm{Sin}^2\phi\left(1+\frac{D}{a^2E'}\right)} + (1+\nu)\,\mathrm{Cot}^2\phi\right]\right\}\lambda^{(n)} \\
& - \{(1-\nu)\,\mathrm{Cot}\phi\}\,M_{\phi\phi} + \left\{\frac{1}{\mathrm{Sin}\phi}\right\}\psi_\phi^{(n)} \\
& \pm \left\{\frac{2nD}{a^2E'\,\mathrm{Sin}^2\phi\left(1+\frac{D}{a^2E'}\right)}\right\}\psi_\theta^{(n)}
\end{aligned}\right\} \quad 4.1.6$$

$$
\begin{aligned}
\frac{d\psi_\phi^{(n)}}{d\phi} = &-\{(1-\nu)E' \cos\phi\,[\,(1+\nu) + \frac{N}{E'} + \frac{n^2 D(1+\nu)}{a^2 E' \sin^2\phi}\,]\}\, u^{(n)} \\
&\mp \{(1-\nu)E' n[\,(1+\nu) + \frac{N}{E'} + \frac{n^2(1+\nu)D}{a^2 E' \sin^2\phi}\,]\}\, v^{(n)} \\
&+ \{\frac{(1-\nu^2)E'}{\sin\phi}\,[\,\sin^2\phi + \frac{n^2 N}{E'(1+\nu)} + \frac{n^4 D}{a^2 E' \sin^2\phi} \\
&+ \frac{2n^2 D \cot^2\phi}{a^2 E'(1+\nu)(1+\frac{D}{a^2E'})}\}\quad w^{(n)} \\
&- \{(1-\nu)\, aE' \cot\phi\,[\,\frac{n^2(1+\nu)D}{a^2 E' \sin\phi} + \frac{2n^2 D}{a^2 E' \sin\phi\,(1+\frac{D}{a^2E'})} \\
&+ \frac{N}{E'}\sin\phi\,]\}\,\lambda^{(n)} - \{(1+\nu)\, a \sin\phi\}\, N_{\phi\phi}^{(n)} \\
&+ \{\frac{N}{D} a^2 \sin\phi + \frac{\nu n^2}{\sin\phi}\}\, M_{\phi\phi}^{(n)} \pm \frac{2nD\cot\phi}{a^2 E' \sin\phi\,(1+\frac{D}{a^2E'})}\}\,\psi_\theta^{(n)} \\
&- \{a^2 \sin\phi\}\, Z^{(n)}
\end{aligned}
\qquad 4.1.7
$$

$$
\left.
\begin{aligned}
\frac{d\psi_\theta}{d\phi} = &\left\{ \frac{1}{1 + \dfrac{2N}{E'(1-\nu)\left(1 + \dfrac{D}{a^2E'}\right)}} \right\} \Big[\pm \left\{ nE'\operatorname{Cot}\phi \left[\frac{\nu N}{E'} \right.\right. \\
&+ (1-\nu^2)\left(1 + \frac{D}{a^2E'}\right)\Big]\Big\} \; u^{(n)} + \frac{n^2E'}{\operatorname{Sin}\phi}\Big[\frac{\nu N}{E'} \\
&+ (1-\nu^2)\left(1 + \frac{D}{a^2E'}\right)\Big]\Big\} \; v^{(n)} \mp \Big\{ nE'\Big[\frac{\nu N}{E'} \\
&+ \frac{2N}{E'} \; \frac{D(1+\nu n^2 - (1+\nu)\operatorname{Sin}\phi)}{a^2E'\operatorname{Sin}^2\phi \left(1 + \dfrac{D}{a^2E'}\right)} + (1-\nu^2)\left(1 + \frac{n^2 D}{a^2E'\operatorname{Sin}^2\phi}\right)\Big]\Big\} \; w^{(n)} \\
&\pm \Big\{ \frac{nD(1+\nu)\operatorname{Cot}\phi}{a} \Big[\frac{2N}{E'\left(1 + \dfrac{D}{a^2E'}\right)} + (1-\nu) \Big]\Big\} \; \lambda^{(n)} \\
&\pm \Big\{ na\Big[\nu - \frac{N}{E'} + \frac{2N}{E'} \; \frac{D}{a^2E'\left(1 + \dfrac{D}{a^2E'}\right)} \Big]\Big\} \; N_{\phi\phi}^{(n)} \\
&\mp \Big\{ n\Big[- \frac{2N}{E'\left(1 + \dfrac{D}{a^2E'}\right)} + \nu \Big]\Big\} \; M_{\phi\phi}^{(n)} - \{\operatorname{Cot}\phi\}\, \psi_\theta^{(n)} \\
&- \{ a^2\operatorname{Sin}\phi \} \; Y^{(n)} \Big]
\end{aligned}
\right\} \quad 4.1.8
$$

$$N_{\theta\theta}^{(n)} = \left\{\frac{(1-\nu^2)E'\,Cot\phi}{a}\right\} u^{(n)} \pm \left\{\frac{n(1-\nu^2)E'}{a\,Sin\,\phi}\right\} v^{(n)} + \{\nu\}\, N_{\phi\phi}^{(n)} - \left\{\frac{(1-\nu^2)E'}{a}\right\} w^{(n)} \tag{4.1.9}$$

$$M_{\theta\theta}^{(n)} = -\left\{\frac{(1-\nu^2)D\,Cot\phi}{a^2}\right\} u^{(n)} \mp \left\{\frac{n(1-\nu^2)D}{a^2\,Sin\,\phi}\right\} v^{(n)} + \left\{\frac{n^2(1-\nu^2)D}{a^2 Sin^2\phi}\right\} w^{(n)} - \left\{\frac{(1-\nu^2)D\,Cot\,\phi}{a}\right\} \lambda^{(n)} + \{\nu\}\, M_{\phi\phi}^{(n)} \tag{4.1.10}$$

$$N_{\theta\phi}^{(n)} = \mp \left\{\frac{n\,D(1-\nu)\,Cot\,\phi}{a^3 Sin\,\phi(1+\frac{D}{a^2E'})}\right\} w^{(n)} \pm \left\{\frac{n\,D(1-\nu)}{a^2\,Sin\,\phi\,(1+\frac{D}{a^2E'})}\right\} \lambda^{(n)} + \left\{\frac{1}{a\,Sin\phi\,(1+\frac{D}{a^2E'})}\right\} \psi_\theta^{(n)} \tag{4.1.11}$$

$$M_{\theta\phi}^{(n)} = \pm \left\{\frac{n\,D(1-\nu)\,Cot\,\phi}{a^2E'\,Sin\phi\,(1+\frac{D}{a^2E'})}\right\} w^{(n)} \mp \left\{\frac{n\,D(1-\nu)}{a\,Sin\,\phi\,(1+\frac{D}{a^2E'})}\right\} \lambda^{(n)} + \left\{\frac{D}{a^2E'\,Sin\phi\,(1+\frac{D}{a^2E'})}\right\} \psi_\theta^{(n)} \tag{4.1.12}$$

$$N_{\phi z}^{(n)} = \left\{\frac{n^2D(1-\nu)\,Cot\,\phi}{a^3Sin^2\phi\,(1+\frac{D}{a^2E'})}\right\} w^{(n)} - \left\{\frac{n^2D(1-\nu)}{a^2Sin^2\phi\,(1+\frac{D}{a^2E'})}\right\} \lambda^{(n)} \pm \left\{\frac{n\,D}{a^3E'\,Sin^2\phi\,(1+\frac{D}{a^2E'})}\right\} \psi_\theta^{(n)} + \left\{\frac{1}{a\,Sin\,\phi}\right\} \psi_\phi^{(n)} \tag{4.1.13}$$

$$
\left.
\begin{aligned}
N_{\theta z}^{(n)} = {} & \pm \left\{ \frac{nD(1-\nu)(\mathrm{Cos}^2\phi + 1)}{a^2 \mathrm{Sin}^3\phi \left(1 + \frac{D}{a^2 E'}\right)} \right\} w^{(n)} \mp \left\{ \frac{2nD(1-\nu)\mathrm{Cot}\phi}{a \mathrm{Sin}\phi \left(1 + \frac{D}{a^2 E'}\right)} \right\} \lambda^{(n)} \\
& + \left\{ \frac{D \,\mathrm{Cot}\,\phi}{a^2 E' \mathrm{Sin}\,\phi \left(1 + \frac{D}{a^2 E'}\right)} \right\} \psi_{\theta}^{(n)} \pm \left\{ \frac{nD(1-\nu)}{a \mathrm{Sin}\phi \left(1 + \frac{D}{a^2 E'}\right)} \right\} \frac{d\lambda^{(n)}}{d\phi} \\
& - \left\{ \frac{D}{a^2 E' \mathrm{Sin}\phi \left(1 + \frac{D}{a^2 E'}\right)} \right\} \frac{d\psi_{\theta}^{(n)}}{d\phi} \mp \left\{ \frac{n}{a \mathrm{Sin}\phi} \right\} M_{\theta\theta}^{(n)} \\
& - \left\{ \frac{2\,\mathrm{Cot}\phi}{a} \right\} M_{\theta\phi}^{(n)}
\end{aligned}
\right\} \quad 4.1.14
$$

4.2 Nondimensionalization of Equations

For engineering problems, it is frequently advantageous to present results in a nondimensional form. This is particularly true where design techniques or general results are involved. The nondimensional form permits one set of data or calculations to be applied to a large number of problems. Examination of the equations 4.1.1 through 4.1.14 shows that the selection of certain quantities as the nondimensionalizing factors will result in some simplification of the equations. Accordingly, equations 4.1.1 through 4.1.14 have been nondimensionalized. These nondimensional equations are presented in Chapter 1 as equations 1.1.1 through 1.1.16.

The scheme of nondimensionalization is presented below. In general "a primed quantity" will denote a nondimensional quantity. There is one exception. This is E', a dimensional quantity as defined previously.

$$\left.\begin{array}{llll}
u^{(n)'} = \dfrac{u^{(n)}}{a} & & v^{(n)'} = \dfrac{v^{(n)}}{a} \\[2ex]
w^{(n)'} = \dfrac{w^{(n)}}{a} & & N_{\phi\phi}^{(n)'} = \dfrac{N_{\phi\phi}^{(n)}}{E'} \\[2ex]
M_{\phi\phi}^{(n)'} = \dfrac{M_{\phi\phi}^{(n)}}{aE'} & & \psi_{\phi}^{(n)'} = \dfrac{\psi_{\phi}^{(n)}}{aE'} \\[2ex]
\psi_{\theta}^{(n)'} = \dfrac{\psi_{\theta}}{aE'} & & X^{(n)'} = \dfrac{a}{E'}X^{(n)} \\[2ex]
Y^{(n)'} = \dfrac{a}{E'}Y^{(n)} & & Z^{(n)'} = \dfrac{a}{E'}Z^{(n)} \\[2ex]
N' = \dfrac{N}{E'} & & D' = \dfrac{D}{a^2E'}
\end{array}\right\} \quad 4.2.1$$

Chapter 5.

Integration of Equations

The solution of the problem for stresses and deflections in a spherical shell has now been reduced to the integration of a set of eight linear first-order differential equations. This set is readily integrated on a digital computer using one of several integration procedures. The program prepared as part of this project utilizes a fourth-order Runge-Kutta process.

The numerical integration presents several mathematical complications, each of which has been resolved in the accompanying program.

The first, and most easily overcome, complication results from the fact that the problems to be solved are boundary value problems whereas numerical integration schemes are directly applicable only to initial value problems. Since the governing differential equations are linear, the boundary value problems can be solved by constructing a linear combination of solutions to judiciously chosen initial value problems so that both the initial and final boundary conditions are met.

The second difficulty occurs when the angles $\phi = 0$ and/or $\theta = \pi$ lie within the region of integration. The differential equations contain singularities at these points. In order to avoid these singularities, such artifices as a small hole, a small rigid plug, or a small elastic plate or cap may be introduced into the program. In the case of the program presented, an extremely small hole subtending a half-angle equal to 10^{-6} times the outside angle is used.

The program may be readily modified to use any of the alternate possibilities which have been mentioned above.